



TITLE:

# Estimates of Convergence Rates for Approximate Solutions of Stochastic Differential Equations(2nd Workshop on Stochastic Numerics)

AUTHOR(S):

KANAGAWA, Shuya

---

CITATION:

KANAGAWA, Shuya. Estimates of Convergence Rates for Approximate Solutions of Stochastic Differential Equations(2nd Workshop on Stochastic Numerics). 数理解析研究所講究録 1995, 932: 125-134

ISSUE DATE:

1995-12

URL:

<http://hdl.handle.net/2433/59970>

RIGHT:

# Estimates of Convergence Rates for Approximate Solutions of Stochastic Differential Equations

Shuya KANAGAWA

Department of Mathematics, Faculty of Liberal Arts & Education,  
Yamanashi University, 4-4-37, Takeda, Kôfu 400, JAPAN,  
e-mail: sgk02122@niftyserve.or.jp

## Abstract

We estimate the error of the Euler-Maruyama type approximate solutions for Itô's stochastic differential equations using the K-M-T inequality. The obtained result can be applied to Monte Carlo simulations of stochastic differential equations.

## Key words

Euler-Maruyama scheme, Stochastic differential equation, Monte Carlo simulation, Pseudo-random numbers, K-M-T inequality

## 1. INTRODUCTION AND RESULTS

Let  $\{B(t), 0 \leq t \leq 1\}$  be an  $r$ -dimensional standard Brownian motion on a probability space  $(\Omega, \mathcal{F}, P)$ . Consider Itô's stochastic differential equation for a  $d$ -dimensional continuous process  $\{X(t), 0 \leq t \leq 1\}$  ( $d \geq 1$ ):

$$(1.1) \quad \begin{cases} dX(t) = \sigma(t, X(t))dB(t) + b(t, X(t))dt, & 0 \leq t \leq 1 \\ X(0) = X_0, \end{cases}$$

where  $\sigma(t, x)$  is a Borel measurable function  $(t, x) \in [0, 1] \times \mathbf{R}^d$

$\rightarrow \mathbf{R}^d \otimes \mathbf{R}^r$  and  $h(t, x)$  is a Borel measurable function  $(t, x) \in [0, 1] \times \mathbf{R}^d \rightarrow \mathbf{R}^d$ . If  $\sigma(t, x)$  and  $h(t, x)$  satisfy the Lipschitz condition, then there exists a unique solution of (1.1). To prove the existence of the unique solution of (1.1) Maruyama [8] constructed an Euler type approximate solution  $Z_n := \{Z_n(t), 0 \leq t \leq 1\}$  defined by

$$Z_n(t) := X_0 + \int_0^t \sigma_n(u) d\mathbf{B}(u) + \int_0^t b_n(u) du, \quad 0 \leq t \leq 1,$$

where

$$\sigma_n(t) := \sigma_n\left(\frac{k-1}{n}, x_{k-1}\right), \quad k/n \leq t \leq (k+1)/n, \quad k=0, \dots, n-1,$$

$$b_n(t) := b_n\left(\frac{k-1}{n}, x_{k-1}\right), \quad k/n \leq t \leq (k+1)/n, \quad k=0, \dots, n-1,$$

$$x_k := X_0 + \sum_{j=1}^k \sigma\left(\frac{j-1}{n}, x_{j-1}\right) \eta_j + \sum_{j=1}^k b\left(\frac{j-1}{n}, x_{j-1}\right) / n, \quad k=0, 1, \dots, n,$$

$$\eta_k := B\left(\frac{k}{n}\right) - B\left(\frac{k-1}{n}\right), \quad k=1, \dots, n..$$

Applying the approximate solution  $Z_n := \{Z_n(t), 0 \leq t \leq 1\}$  to Monte Carlo simulations on digital computers we must discretize it as following. Let  $X_n := \{X_n(t), 0 \leq t \leq 1\}$  be a stochastic process in  $D[0, 1]$  defined by

$$\begin{cases} X_n(t) := x_k, & k/n \leq t < (k+1)/n, \quad k=0, \dots, n-1 \\ X_n(1) := x_n, \end{cases}$$

where  $\{\eta_k\}$  are i.i.d. random variables with the  $r$ -dimensional normal distribution  $N(0, 1/n)$  which are realized by pseudo-normal random numbers. As for the error estimation for  $X_n$  and  $Z_n$ , Ghiman-Skorohod [1] and Kanagawa [2]-[4] showed the rate of convergence of them to the real solution  $X$  of (1.1) in  $L^p$ -mean

for some  $p \geq 2$ . Ogawa [7] estimated the error of Euler-Maruyama approximate solutions of some nonlinear diffusion processes governed by Itô's stochastic differential equation. For more details of other types of approximate solutions, see also Kloeden-Platen [5].

Since pseudo-uniform random numbers are generated by some algebraic algorithms, it is obvious that the distribution of them is not the real uniform distribution. Thus pseudo-normal random numbers, which are generated from pseudo-uniform random numbers by some methods, e.g. the Box-Muller method, do not obey the real normal distribution, too. Furthermore, as for the speed of computation by digital computers, approximate solutions constructed from pseudo-uniform random numbers have advantage over the construction from pseudo-normal random numbers. Therefore we shall estimate the convergence rate of the approximate solutions to the real solution of (1.1) when the distribution of  $\{\eta_k\}$  is not the normal distribution. In [3] the rate of convergence is considered assuming the existence of the third absolute moment for  $\{\eta_k\}$ . Let  $\{\xi_k\}$  be  $r$ -dimensional i.i.d. random variables which do not always obey the normal distribution and define an approximate solution  $\{Y_n(t), 0 \leq t \leq 1\}$  by

$$\begin{cases} Y_n(t) := y_k, & k/n \leq t < (k+1)/n, \quad k=0, \dots, n-1 \\ Y_n(1) := y_n, \end{cases}$$

here

$$y_k := X_0 + \sum_{j=1}^k \sigma\left(\frac{j-1}{n}, y_{j-1}\right) \xi_j / \sqrt{n} + \sum_{j=1}^k b\left(\frac{j-1}{n}, y_{j-1}\right) / n, \quad k=0, 1, \dots, n.$$

**Theorem A.** ([3]) Let  $\{\xi_k, k \geq 1\}$  be  $r$ -dimensional i.i.d. random variables with

$$(1.2) \quad E(\xi_1) = 0, E(|\xi_1|^2) = 1 \text{ and } E(|\xi_1|^3) < \infty.$$

Suppose that for any  $0 \leq s, t \leq 1$  and  $x, y \in \mathbf{R}^d$

$$(1.3) \quad \left| \sigma(t, x) - \sigma(s, y) \right|^2 + \left| b(t, x) - b(s, y) \right|^2 \leq K_1 \left( \left| x - y \right|^2 + \left| t - s \right|^2 \right),$$

$$(1.4) \quad \left| \sigma(t, x) \right|^2 + \left| b(s, y) \right|^2 \leq K_2,$$

where  $K_1$  and  $K_2$  are some positive constants independent of  $s, t, x$  and  $y$ . Then we can redefine  $\{X(t), 0 \leq t \leq 1\}$  and  $\{Y_n(t), 0 \leq t \leq 1\}$  on a common probability space such that for any  $p \geq 2$  and  $\varepsilon > (2 + \delta)^2 / 2(3 + \delta)$ ,

$$(1.5) \quad E \left( \max_{0 \leq t \leq 1} \left| X(t) - Y_n(t) \right|^p \right) = o \left( n^{-p/6} (\log n)^\varepsilon \right) \text{ as } n \rightarrow \infty,$$

where the power of  $n$  cannot be improved by better one.

The aim of this paper is to improve the above result under the Cramér condition which is satisfied several types of distributions, e.g. the uniform distribution.

**Theorem 1.** Let  $\{\xi_k, k \geq 1\}$  be  $r$ -dimensional i.i.d. random variables with zero mean and finite variance. Moreover let  $\{\xi_k, k \geq 1\}$  satisfy the Cramér condition;

$$(1.6) \quad E \left( \exp(s|\xi_1|) \right) < \infty \text{ in a neighborhood of } s = 0.$$

Assume  $\sigma(t, x)$  and  $b(t, x)$  satisfy (1.3) and (1.4). Then we can redefine  $\{X(t), 0 \leq t \leq 1\}$  and  $\{Y_n(t), 0 \leq t \leq 1\}$  on a common probability space such that for any  $p \geq 2$  and  $\varepsilon > p$ ,

$$(1.7) \quad E \left( \max_{0 \leq t \leq 1} \left| X(t) - Y_n(t) \right|^p \right) = o \left( n^{-p/4} (\log n)^\varepsilon \right) \text{ as } n \rightarrow \infty.$$

## 2. PRELIMINARIES

Before proving the theorem we define two random processes  $\{\bar{X}_n(t), 0 \leq t \leq 1\}$  and  $\{\bar{Y}_n(t), 0 \leq t \leq 1\}$  as follows. Let  $\{\zeta_k, 1 \leq k \leq M\}$  and  $\{\eta_k, 1 \leq k \leq M\}$  be random variables defined by

$$\zeta_k = \sum_{i=(k-1)[n^{1/2}]+1}^{k[n^{1/2}]} \frac{\xi_i}{\sqrt{n}}, \quad 1 \leq k \leq M-1, \quad \zeta_M = \sum_{i=(M-1)[n^{1/2}]+1}^n \frac{\xi_i}{\sqrt{n}}$$

$$\eta_k = B(k[n^{1/2}]/n) - B(((k-1)[n^{1/2}]+1)/n), \quad 1 \leq k \leq M-1,$$

and

$$\eta_M = B(1) - B(((M-1)[n^{1/2}]+1)/n),$$

where  $M = [n/[n^{1/2}]] + 1$ . Define  $\{\bar{X}_n(t), 0 \leq t \leq 1\}$  and  $\{\bar{Y}_n(t), 0 \leq t \leq 1\}$  by

$$\begin{cases} \bar{X}_n(t) := u_k, & ((k-1)[n^{1/2}]+1)/n \leq t < k[n^{1/2}]/n, \quad 1 \leq k \leq M-1 \\ \bar{X}_n(1) := u_M, \\ \bar{Y}_n(t) := v_k, & ((k-1)[n^{1/2}]+1)/n \leq t < k[n^{1/2}]/n, \quad 1 \leq k \leq M-1 \\ \bar{Y}_n(1) := v_M, \end{cases}$$

where

$$u_k := X_0 + \sum_{j=1}^k \sigma((j-1)[n^{1/2}]/n, u_{j-1}) \eta_j + \sum_{j=1}^k b((j-1)[n^{1/2}]/n, u_{j-1}) [n^{1/2}]/n, \quad 1 \leq k \leq M,$$

$$v_k := X_0 + \sum_{j=1}^k \sigma((j-1)[n^{1/2}]/n, v_{j-1}) \zeta_j + \sum_{j=1}^k b((j-1)[n^{1/2}]/n, v_{j-1}) [n^{1/2}]/n, \quad 1 \leq k \leq M.$$

The following result, which is called the K-M-T inequality obtained by Komlós-Major-Tusnády [6], plays an important roll to estimate  $E(|\eta_1 - \zeta_1|^2)$ .

**Lemma 1.** *Given the condition (1.6), there exist a Brownian motion  $\{B(t), 0 \leq t \leq 1\}$  and  $\{\xi_k, k \geq 1\}$  on a common probability space such that for all*

real  $x$  and every  $n$  we have

$$(2.1) \quad P \left\{ \max_{1 \leq k \leq n} \left| \sum_{i=1}^k \xi_i - B(k) \right| \geq C \log n + x \right\} < K e^{-\lambda x},$$

where  $C, K, \lambda$  are positive constants depending only on the distribution function of  $\xi_1$ .

Using the K-M-T inequality we have the next lemma.

**Lemma 2.** Without changing distributions of  $\{\xi_k, 1 \leq k \leq n\}$  and  $\{\zeta_k, 1 \leq k \leq M\}$ , we can redefine them on a richer probability space with a Brownian motion  $\{B(t), 0 \leq t \leq 1\}$  with the increments  $\{\eta_k, 1 \leq k \leq M\}$  such that for each  $1 \leq k \leq M$  and for any  $\varepsilon > 2$

$$(2.2) \quad E(|\zeta_k - \eta_k|^2) = o(n^{-1}(\log n)^\varepsilon) \text{ as } n \rightarrow \infty,$$

$$(2.3) \quad \{\eta_1, \dots, \eta_k, \xi_1, \dots, \xi_k\} \text{ is independent of } \{\eta_{k+1}, \dots, \eta_M, \xi_{k+1}, \dots, \xi_M\}.$$

**Proof.** By (2.1) we have

$$\begin{aligned} nE(|\zeta_1 - \eta_1|^2) &= \int \left\{ \left| \sum_{i=1}^{[n^{1/n}]} \xi_i - B([n^{1/n}]/n) \right| \geq C \log n + \frac{4}{\lambda} \log n \right\} \left| \sum_{i=1}^{[n^{1/n}]} \xi_i - B([n^{1/n}]/n) \right|^2 dP \\ &\quad + \int \left\{ \left| \sum_{i=1}^{[n^{1/n}]} \xi_i - B([n^{1/n}]/n) \right| < C \log n + \frac{4}{\lambda} \log n \right\} \left| \sum_{i=1}^{[n^{1/n}]} \xi_i - B([n^{1/n}]/n) \right|^2 dP \\ &\leq \sum_{k=0}^{\infty} \int \left\{ C \log n + \frac{4}{\lambda} \log n \leq \left| \sum_{i=1}^{[n^{1/n}]} \xi_i - B([n^{1/n}]/n) \right| \leq C \log n + \frac{4}{\lambda} \log n + k \right\} \left| \sum_{i=1}^{[n^{1/n}]} \xi_i - B([n^{1/n}]/n) \right|^2 dP \end{aligned}$$

$$\begin{aligned}
& + \left( C \log n + \frac{4}{\lambda} \log n \right)^2 \mathbf{P} \left\{ \left| \sum_{i=1}^{\lfloor n^{1/n} \rfloor} \xi_k - B(\lfloor n^{1/n} \rfloor / n) \right| < C \log n + \frac{4}{\lambda} \log n \right\} \\
& \leq \sum_{k=0}^{\infty} \left( C \log n + \frac{4}{\lambda} \log n + k \right)^2 \mathbf{P} \left\{ \left| \sum_{i=1}^{\lfloor n^{1/n} \rfloor} \xi_k - B(\lfloor n^{1/n} \rfloor / n) \right| \geq C \log n + \frac{4}{\lambda} \log n + k \right\} \\
& \quad + \left( C \log n + \frac{4}{\lambda} \log n \right)^2 \\
& \leq K \sum_{k=0}^{\infty} \left( C \log n + \frac{4}{\lambda} \log n + k \right)^2 \exp \left\{ -\lambda \left( \frac{4}{\lambda} \log n + k \right) \right\} + \left( C \log n + \frac{4}{\lambda} \log n \right)^2 \\
& = O((\log n)^2).
\end{aligned}$$

Moreover  $E(|\zeta_k - \eta_k|^2), k \geq 2$ , are treated similarly. On the other hand, taking independent copies of  $(\zeta_1, \eta_1)$ , (2.3) can be shown easily.  $\square$

### 3. PROOF OF THEOREM 1

For simplicity we treat the case  $d=r=1$ . The multidimensional case can be proved similarly. In what follows  $K$ 's are different positive constants independent of  $n$  in different equations. From the definitions of  $X_n(t)$  and  $Y_n(t)$

$$\begin{aligned}
(3.1) \quad E \left( \max_{0 \leq u \leq t} |X(u) - Y_n(u)|^p \right)^{1/p} & \leq E \left( \max_{0 \leq u \leq t} |X(u) - \bar{X}_n(u)|^p \right)^{1/p} \\
& \quad + E \left( \max_{0 \leq u \leq t} |\bar{X}_n(u) - \bar{Y}_n(u)|^p \right)^{1/p} + E \left( \max_{0 \leq u \leq t} |\bar{Y}_n(u) - Y_n(u)|^p \right)^{1/p} \\
& = I_1^{1/p} + I_2^{1/p} + I_3^{1/p}.
\end{aligned}$$



We first estimate  $I_2$ . For  $\frac{k[n^{1/2}]}{n} \leq t \leq \frac{(k+1)[n^{1/2}]}{n}$ ,  $1 \leq k \leq M$ , we have from (1.3)

$$\begin{aligned}
 (3.2) \quad I_2 &\leq E \left( \max_{1 \leq i \leq k} |u_i - v_i|^p \right) \\
 &\leq K E \left( \max_{1 \leq i \leq k} \left| \sum_{j=1}^i \sigma((j-1)[n^{1/2}]/n, u_{j-1}) \eta_j - \sum_{j=1}^i \sigma((j-1)[n^{1/2}]/n, v_{j-1}) \zeta_j \right|^p \right) \\
 &\quad + K E \left( \max_{1 \leq i \leq k} \left| \sum_{j=1}^i b((j-1)[n^{1/2}]/n, u_{j-1}) [n^{1/2}]/n - \sum_{j=1}^i b((j-1)[n^{1/2}]/n, v_{j-1}) [n^{1/2}]/n \right|^p \right) \\
 &\leq K E \left( \max_{1 \leq i \leq k} \left| \sum_{j=1}^i \sigma((j-1)[n^{1/2}]/n, u_{j-1}) (\eta_j - \zeta_j) \right|^p \right) \\
 &\quad + K E \left( \max_{1 \leq i \leq k} \left| \sum_{j=1}^i (\sigma((j-1)[n^{1/2}]/n, u_{j-1}) - \sigma((j-1)[n^{1/2}]/n, v_{j-1})) \zeta_j \right|^p \right) \\
 &\quad + K E \left( \max_{1 \leq i \leq k} \left| \sum_{j=1}^i (b((j-1)[n^{1/2}]/n, u_{j-1}) - b((j-1)[n^{1/2}]/n, v_{j-1})) [n^{1/2}]/n \right|^p \right) \\
 &=: I_{21} + I_{22} + I_{23}.
 \end{aligned}$$

Put

$$S_k := \sum_{j=1}^k \sigma((j-1)[n^{1/2}]/n, u_{j-1}) (\eta_j - \zeta_j), \quad 1 \leq k \leq M.$$

Since (2.3) implies that  $\{S_k, k \geq 1\}$  is a  $\mathcal{F}_k := \sigma\{\eta_1, \dots, \eta_k, \xi_1, \dots, \xi_k\}$ -adapted

martingale, using Doob's inequality and (1.4), we have

$$\begin{aligned}
 (3.3) \quad I_{21} &= K E \left( \max_{1 \leq i \leq k} |S_i|^p \right) \\
 &\leq K \left\{ E \left( \sum_{j=1}^k E \left( \left| \sigma((j-1)[n^{1/2}]/n, u_{j-1}) (\eta_j - \zeta_j) \right|^2 \mid \mathcal{F}_{j-1} \right) \right) \right\}^{p/2} \\
 &\leq K \left\{ \sum_{j=1}^k E \left( |\eta_j - \zeta_j|^2 \right) \right\}^{p/2}.
 \end{aligned}$$

Therefore we obtain from (2.2) and (3.3) that

$$(3.4) \quad I_{22} \leq K t^{p/2} n^{-p/4} (\log n)^p.$$

On the other hand by Doob's inequality it is easy to see that

$$(3.5) \quad I_{21} \leq K \int_0^t E \left( \max_{0 \leq u \leq s} |\bar{X}_n(u) - \bar{Y}_n(u)|^p \right) ds,$$

$$(3.6) \quad I_{23} \leq K \int_0^t E \left( \max_{0 \leq u \leq s} |\bar{X}_n(u) - \bar{Y}_n(u)|^p \right) ds.$$

From (3.1)-(3.6), for any  $0 \leq t \leq 1$

$$E \left( \max_{0 \leq u \leq t} |\bar{X}_n(u) - \bar{Y}_n(u)|^p \right) \leq K t^{p/2} n^{-p/4} (\log n)^p + K \int_0^t E \left( \max_{0 \leq u \leq s} |\bar{X}_n(u) - \bar{Y}_n(u)|^p \right) ds.$$

Thus by Gronwall's lemma

$$(3.7) \quad I_2 = o(n^{-p/4} (\log n)^\varepsilon),$$

as  $n \rightarrow \infty$ , for any  $\varepsilon > p$ . As for  $I_1$  and  $I_3$ , from Lemmas 5 and 6 in [3], we have as  $n \rightarrow \infty$

$$(3.8) \quad I_1 = O(n^{-p/4}) \text{ and } I_3 = O(n^{-p/4}).$$

From (3.1), (3.7) and (3.8) we conclude the proof of the theorem.  $\square$

## REFERENCES

- [1] Gihman, I. I. and Skorohod, A. V., The Theory of Stochastic Processes III, Springer-Verlag, Berlin, 1979.
- [2] Kanagawa, S. (1988), On the rate of convergence for Maruyama's approximate solutions of stochastic differential equations, Yokohama Math. J., 36, 79-85.
- [3] Kanagawa, S. (1989), The rate of convergence for approximate solutions of stochastic differential equations, Tokyo J. Math., 12, 33-48.
- [4] Kanagawa, S. (1995), Error estimation for the Euler-Maruyama approximate solutions of stochastic differential equations, Monte Carlo Methods and Applications,
- [5] Kloeden, P. E. and Platen, E., Numerical Solutions of Stochastic Differential Equations, Springer-Verlag, Berlin, 1992.
- [6] Komlós, J., Major, J. and Tusnády, G. (1975), An approximation of partial sums of independent R. V.'s and the sample DF. I, Z. Wahrscheinlichkeitstheorie verw. Gebiete, 32, 111-131.
- [7] Ogawa, S. (1992), Monte Carlo Simulation of Nonlinear Diffusion processes, Japan J. of Industrial and Appl. Math., 9, 22-33.
- [8] Maruyama, G. (1955), Continuous Markov processes and stochastic equations, Rend Circ. Mat. Palermo, 4, 48-90.